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LETTER TO THE EDITOR

Exceptional structure of the dilute A_3 model: E_8 and E_7 Rogers–Ramanujan identities

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Abstract. *The dilute A_3 lattice model in regime 2 is in the universality class of the Ising model* in a magnetic field. Here we directly establish the existence of an E_8 structure in the dilute A_3 model in this regime by expressing the one-dimensional configuration sums in terms of fermionic sums which involve the E_8 root system explicitly. In the thermodynamic limit, these polynomial identities yield a proof of the E_8 Rogers–Ramanujan identity recently conjectured by Kedem *et al.* The polynomial identities also apply to regime 3, which is obtained by transforming the modular parameter by $q \to 1/q$. In this case we find an $A_1 \times E_7$ structure and prove a Rogers–Ramanujan identity of $A_1 \times E_7$-type. Finally, in the critical $q \to 1$ limit, we give some intriguing expressions for the number of L-step paths on the A_3 Dynkin diagram with tadpoles in terms of the E_8 Cartan matrix. All our findings confirm the E_8 and E_7 structure of the dilute A_3 model found recently by means of the thermodynamic Bethe ansatz.

Recently, a Bethe ansatz study [1] of the dilute A_3 lattice model [2, 3] has revealed a hidden E_8 structure. This establishes the expected relation between the dilute A_3 model and Zamolodchikov's E_8 S-matrix of the critical Ising model in a field [4]. One of the drawbacks, however, of this Bethe ansatz approach is that it relies heavily on the acceptance of a conjectured string structure of the Bethe ansatz equations. In this letter we demonstrate the E_8 structure of the dilute A_3 model directly, without the use of a string hypothesis.

In recent papers Melzer [5] and Berkovich [6] have shown that the one-dimensional configuration sums of the ABF model admit a so-called fermionic representation in addition to the well known bosonic forms of Andrews *et al* [7]. Their motivation was, in fact, to prove Rogers–Ramanujan (RR) type identities for the $\chi^{(h)}_{r,s}$ Virasoro characters associated with the minimal unitary models of central charge $c = 1 - 6/h(h-1)$ as conjectured by the Stony Brook group [8]. In this letter we adopt a similar approach. Specifically, we rewrite the known bosonic expressions for the one-dimensional configuration sums of the dilute A_3 model in the 2^+ regime [3] in terms of fermionic sums. These fermionic sums explicitly involve the E_8 root system. In particular, in the thermodynamic limit, our 'fermionic sum = bosonic sum' expressions yield precisely the E_8 Rogers–Ramanujan identity for the $\chi^{(4)}_{1,1}$ Virasoro character as given by Kedem *et al* [9].

Thermodynamic Bethe ansatz computations were also carried out [1] in the 3^+ regime of the dilute A_3 model. In this case the model is known [3] to decouple, in the scaling limit, into an Ising model and a $\phi^{(5)}_{2,1}$ perturbed minimal model. Accordingly, the Bethe ansatz

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computations [1, 10] on the dilute A_3 model in the 3^+ regime gives an $A_1 \times E_7$ structure leading to the correct central charge $c = \frac{1}{2} + \frac{7}{10} = \frac{6}{5}$. These findings for the 3^+ regime are supported in this letter. By considering the thermodynamic limit of our 'fermionic sum = bosonic sum' expressions, after carrying out the transformation $q \to 1/q$, which maps the 2^+ regime onto the 3^+ regime [3], we find an $A_1 \times E_7$ Rogers–Ramanujan identity similar to the E_7 identity for the $\chi^{(5)}_{1,1}$ character conjectured by Kedem *et al* [9].

To conclude this letter, we point out some intriguing expressions for the number of walks on the adjacency graph of the dilute A_3 model in terms of the E_8 Cartan matrix.

Polynomial E_8 Rogers–Ramanujan identity

Before we present the main results of this letter we need to introduce some notation.

We define the Gaussian multinomials or q-multinomials by [11]

$$\begin{bmatrix} N \\ m_1, m_2, \ldots, m_n \end{bmatrix}_q = \frac{(q)_N}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_n}(q)_{N-m_1-m_2-\cdots-m_n}} \tag{1}$$

where $(q)_m = \prod_{k=1}^{m}(1-q^k)$ for $m > 0$ and $(q)_0 = 1$. Also, if $\mathcal{I}_{E_8}$ denotes the incidence matrix of E_8 with the nodes labelled as in figure 1(*a*), we define the following thermodynamic Bethe ansatz (TBA) type systems:

$$\boldsymbol{n} + \boldsymbol{m} = \tfrac{1}{2}(\mathcal{I}_{E_8}\boldsymbol{m} + (L-1)\boldsymbol{e}_1 + \boldsymbol{e}_i) \qquad i = 1, 2, \ldots, 8\,. \tag{2}$$

Here $\boldsymbol{n}$ and $\boldsymbol{m}$ are eight-dimensional column vectors with *integer* entries n_i, m_i, respectively, and $\boldsymbol{e}_i$ is a unit vector with components $(\boldsymbol{e}_i)_j = \delta_{i,j}$. The parameter L will be referred to as the system size. A pair of vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ solving the ith TBA-type equation with system size L will be denoted by $(\boldsymbol{n}, \boldsymbol{m})_{L,i}$. Following [6], we now define the fermionic functions $F_i(L)$ by

$$F_i(L) = \sum_{(\boldsymbol{n},\boldsymbol{m})_{L,i}} q^{\boldsymbol{n}^T C_{E_8}^{-1}\boldsymbol{n}} \prod_{j=1}^{8} \begin{bmatrix} n_j + m_j \\ n_j \end{bmatrix}_q \tag{3}$$

with C_{E_8} the Cartan matrix of E_8 which is related to the incidence matrix by $(C_{E_8})_{i,j} = 2\delta_{i,j} - (\mathcal{I}_{E_8})_{i,j}$. Finally, we define the bosonic functions $B_{r,s}(L, a, b)$ by [3]

$$\begin{aligned} B_{r,s}(L,a,b) = \sum_{j,k=-\infty}^{\infty} \Bigg\{ & q^{12j^2+(4r-3s)j+k(k+8j+b-a)} \begin{bmatrix} L \\ k, k+8j+b-a \end{bmatrix}_q \\ & - q^{12j^2+(4r+3s)j+rs+k(k+8j+b+a)} \begin{bmatrix} L \\ k, k+8j+b+a \end{bmatrix}_q \Bigg\}. \end{aligned} \tag{4}$$

With these definitions our main assertion can be written as

$$F_1(L) = B_{1,1}(L, 1, 1)\,. \tag{5}$$

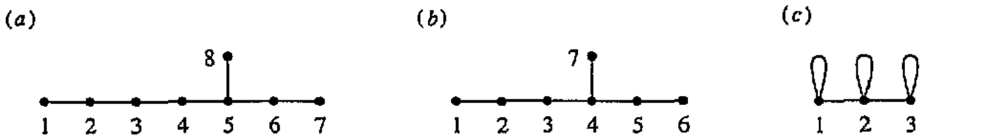

Figure 1. (*a*) The Dynkin diagram of E_8; (*b*) the Dynkin diagram of E_7, and (*c*) the incidence or adjacency graph of the dilute A_3 model.

Explicitly, this polynomial identity takes the form

$$\sum_{(n,m)_{L,1}} q^{n^T C_{E_8}^{-1} n} \prod_{i=1}^{8} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}_q = \sum_{j,k=-\infty}^{\infty} \left\{ q^{12j^2+j+k(k+8j)} \begin{bmatrix} L \\ k, k+8j \end{bmatrix}_q \right.$$
$$\left. - q^{12j^2+7j+1+k(k+8j+2)} \begin{bmatrix} L \\ k, k+8j+2 \end{bmatrix}_q \right\} \tag{6}$$

which can be viewed as a finitization of the E_8 Rogers–Ramanujan identity of Kedem *et al* [9]. Indeed, taking the limit $L \to \infty$, using the result [3]

$$\lim_{L\to\infty} \sum_{k=-\infty}^{\infty} q^{k(k+a)} \begin{bmatrix} L \\ k, k+a \end{bmatrix}_q = \frac{1}{(q)_\infty} \tag{7}$$

together with the simple formula $\lim_{N\to\infty} \left[{N \atop m} \right]_q = 1/(q)_m$, gives

$$\sum_{n_1,\ldots,n_8=0}^{\infty} \frac{q^{n^T C_{E_8}^{-1} n}}{(q)_{n_1} \cdots (q)_{n_8}} = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left\{ q^{12j^2+j} - q^{12j^2+7j+1} \right\}. \tag{8}$$

The RHS of this E_8 Rogers–Ramanujan identity is the usual (bosonic) Rocha-Caridi form for the $\chi_{1,1}^{(4)}$ Virasoro character [12]. The LHS is the fermionic counterpart conjectured by Kedem *et al* [9].

Before we proceed to sketch a proof of identity (6), let us first explain how the above results relate to the E_8 structure of the dilute A_3 model. To this end we note that the bosonic side of (6) is precisely the same expression as for the one-dimensional configuration sum $Y_L^{1\,1\,1}(q)$ of the dilute A_3 model in the 2^+ regime as computed in [3]. More generally, the configuration sums $Y_L^{a\,b\,c}(q)$ with $a, b, c \in \{1, 2, 3\}$ and $|b - c| \leqslant 1$ are defined via

$$Y_L^{\sigma_1\,\sigma_{L+1}\,\sigma_{L+2}}(q) = \sum_{\sigma_2,\ldots,\sigma_L} q^{\sum_{j=1}^{L} j H(\sigma_j, \sigma_{j+1}, \sigma_{j+2})}. \tag{9}$$

The function H follows directly from the Boltzmann weights of the dilute A_3 model by computing the ordered infinite-field limit ($p \to 1$, u/ϵ fixed)

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} \to \frac{g_a g_c}{g_b g_d}\, \mathrm{e}^{-2\pi u H(d,a,b)/\epsilon}\, \delta_{a,c} \qquad \text{with } g_a = \mathrm{e}^{-2\lambda u a^2/\epsilon}. \tag{10}$$

Here u is the spectral parameter, $3\lambda = 15\pi/16$ is the crossing parameter and $p = \exp(-\epsilon)$ is the name of the elliptic function parametrization of the face weights [3]. A complete listing of the values of $H(a, b, c)$ is given in (A.5) of [3]. The occurrence of the particular configuration sum $Y_L^{1\,1\,1}(q)$ in (6) can be understood from the fact that the configuration with all spins on the lattice taking the value 1 corresponds to the ground state of the model.

Sketch of a proof of (6). The proof of identity (6) is long and tedious so we will present it in full elsewhere. Here we only indicate the main ingredients of the proof and omit the detailed calculations.

Let us recall that the configuration sums $Y_L^{a\,b\,c}(q)$ satisfy the recurrences [3]

$$Y_L^{a\,b\,c}(q) = q^{L\,H(b-1,b,c)}\, Y_{L-1}^{a\,b-1\,b}(q) + q^{L\,H(b,b,c)}\, Y_{L-1}^{a\,b\,b}(q) + q^{L\,H(b+1,b,c)}\, Y_{L-1}^{a\,b+1\,b}(q) \tag{11}$$

subject to the initial condition

$$Y_1^{a\,b\,c}(q) = q^{H(a,b,c)}. \tag{12}$$

Moreover, the recurrence relations together with the initial conditions uniquely determine the configuration sums $Y_L^{a\,b\,c}(q)$. In view of (6) we will only consider the case $a = 1$.

Apart from $Y_L^{111}(q)$ we now also express the other $Y_L^{1bc}(q)$ firstly in terms of fermionic sums and secondly in terms of bosonic sums. If we can then show that both the fermionic and bosonic expressions satisfy the recurrences (11) together with (12), the fermionic sums must equal the bosonic ones due to the uniqueness of the solution to (11) and (12).

First, we list the bosonic expressions for $Y_L^{1bc}(q)$:

$$\begin{aligned}
&Y_L^{111}(q) = B_{1,1}(L,1,1) \qquad Y_L^{112}(q) = q^L B_{1,1}(L,1,1) \qquad Y_L^{121}(q) = B_{3,1}(L,1,2)\\
&Y_L^{122}(q) = q^{L-1} B_{1,1}(L,1,2) + q^L\left(1-q^L\right)B_{3,1}(L-1,1,3)\\
&Y_L^{123}(q) = q^{2L-1} B_{1,1}(L,1,2) \qquad Y_L^{132}(q) = q^{-L-1} B_{3,1}(L,1,3)\\
&Y_L^{133}(q) = q^{L-1} B_{3,1}(L,1,3) + q^{L-3}\left(1-q^L\right)B_{1,1}(L-1,1,2)\,.
\end{aligned} \tag{13}$$

The proof that this solves (11) and (12) has been given in [3].

Second, we list fermionic expressions for $Y_L^{1bc}(q)$:

$$\begin{aligned}
Y_L^{111}(q) &= F_1(L)\\
Y_L^{112}(q) &= q^L F_1(L)\\
Y_L^{121}(q) &= \big(F_7(L) - \left(1-q^L\right)F_1(L) - q^L\left(1-q^L\right)F_1(L-1)\\
&\qquad +q^{2L-1}\left(1-q^{L-1}\right)F_1(L-2)\big)/q^{L+1}\\
Y_L^{122}(q) &= \big(F_7(L) - \left(1-q^L\right)F_1(L) + q^{2L-1}\left(1-q^{L-1}\right)F_1(L-2)\big)/q\\
Y_L^{123}(q) &= q^{2L-1}\left(F_7(L) + q^{L-1}\left(1-q^{L-1}\right)F_1(L-2)\right)\\
Y_L^{132}(q) &= \big(F_7(L) - F_1(L) + q^L F_7(L-1) + q^{L-1}\left(1-q^{L-1}\right)F_7(L-2)\\
&\qquad +q^{2L-2}\left(1-q^{L-2}\right)F_1(L-3) + q^{2L-4}\left(1-q^{L-1}\right)\left(1-q^{L-3}\right)F_1(L-4)\big)/q^{L+3}\\
Y_L^{133}(q) &= q^{L-3}\big(F_7(L) - F_1(L) + F_7(L-1) + q^{L-1}\left(1-q^{L-1}\right)F_7(L-2)\\
&\qquad +q^{L-2}\left(1-q^{L-2}\right)F_1(L-3) + q^{2L-4}\left(1-q^{L-1}\right)\left(1-q^{L-3}\right)F_1(L-4)\big)\,.
\end{aligned} \tag{14}$$

These expressions are admittedly quite complicated and might very well be simplified. For example, we have chosen to express all configuration sums in terms of F_1 and F_7 only. Using easily verifiable recurrences of the type $F_1(L) - q^{2L-2}F_1(L-2) = F_2(L-1)$, $F_2(L) + q^{L+1}\left(1-q^{L-2}\right)F_2(L-2) = F_3(L-1) + q^{L+1}F_1(L-1)$, etc, one could conceivably find simpler forms for the above.

To prove the correctness of the fermionic solution to the recurrence relations we substitute (14) into (11). This gives seven identities which can be combined to yield the following two equations:

$$\begin{aligned}
&F_1(L) - F_7(L-1) - q^{L-1}F_1(L-1) + q^{L-1}\left(1-q^{L-1}\right)F_1(L-2)\\
&\qquad -q^{2L-3}\left(1-q^{L-2}\right)F_1(L-3) = 0\\
&F_7(L) - q^2F_1(L-1) - \left(1+q^{L-1}\right)F_7(L-1) + q^2\left(1-q^{2L-4}\right)F_7(L-2)\\
&\qquad +q^{L-1}F_1(L-2) - q^L\left(1-q^{L-2}\right)F_1(L-3)\\
&\qquad +q^L\left(1-q^{L-2}\right)\left(1-q^{L-3}\right)F_7(L-3) = 0\,.
\end{aligned} \tag{15}$$

The actual proof of these final two equations will be omitted here. We remark that they follow from elementary but tedious computations very similar to those carried out in [6]. The proof that (14) satisfies the initial conditions is a matter of straightforward case-checking.

$A_1 \times E_7$ Rogers–Ramanujan identity

We now consider the $L \to \infty$ limit of identity (6) after first replacing q by $1/q$. The effect of this transformation on q is to map from the 2^+ regime to the 3^+ regime of the dilute A_3 model. The critical behaviour of the model in this latter regime is described by a $c = \frac{6}{5}$ [3, 1] conformal field theory given as a direct product of an Ising model ($c = \frac{1}{2}$) and an E_7 theory with $c = 2 \text{ rank } \mathcal{G}/(g+2) = (2)(7)/(18+2) = \frac{7}{10}$. Hence it is to be expected that the above steps will result in an $A_1 \times E_7$ Rogers–Ramanujan identity.

To establish this we use two simple inversion formulae:

$$\begin{bmatrix} N \\ m \end{bmatrix}_{1/q} = q^{m(m-N)} \begin{bmatrix} N \\ m \end{bmatrix}_q \tag{16}$$

$$\begin{bmatrix} N \\ m_1, m_2 \end{bmatrix}_{1/q} = q^{m_1^2+m_2^2+m_1m_2-(m_1+m_2)N} \begin{bmatrix} N \\ m_1, m_2 \end{bmatrix}_q . \tag{17}$$

Applying (17) to transform the bosonic RHS of (6) we obtain, after performing a shift on the summation variable k,

$$q^{(\mu-L^2)/2} \sum_{j,k=-\infty}^{\infty} q^{2k(k+\mu)} \left\{ q^{20j^2+j} \begin{bmatrix} L \\ (L-\mu)/2-4j-k, (L-\mu)/2+4j-k \end{bmatrix}_q \right. \\ \left. -q^{20j^2+9j+1} \begin{bmatrix} L \\ (L-\mu)/2-4j-k-1, (L-\mu)/2+4j-k+1 \end{bmatrix}_q \right\} . \tag{18}$$

Here the variable $\mu = 0$ and 1 is given by the parity of L via $(L-\mu)/2 \in \mathbb{Z}$. Multiplying by the factor $q^{L^2/2}$ and taking the thermodynamic limit using the result

$$\lim_{N\to\infty} \begin{bmatrix} 2N \\ N-a, N-b \end{bmatrix}_q = \frac{1}{(q)_\infty (q)_{a+b}} \qquad a+b \geqslant 0 \tag{19}$$

yields

$$q^{\mu/2} \sum_{k=0}^{\infty} \frac{q^{2k(k+\mu)}}{(q)_{2k+\mu}} \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \{ q^{20j^2+j} - q^{20j^2+9j+1} \} = q^{1/48+7/240} \chi^{(4)}_{1+\mu,1}(q)\, \chi^{(5)}_{1,1}(q) . \tag{20}$$

This final expression has the expected factorized form as alluded to before, with the second term being the $\chi^{(5)}_{1,1}$ character corresponding to a $c = \frac{7}{10}$ conformal field theory.

We now turn to the fermionic LHS of (6). After replacing q with $1/q$, applying (16) and multiplying by the factor $q^{L^2/2}$, the fermionic sum takes the form

$$\sum_{\boldsymbol{n}} q^{(\boldsymbol{n}-L\,\boldsymbol{e}_1/2)^T C^{-1}_{E_8} (\boldsymbol{n}-L\,\boldsymbol{e}_1/2)} \prod_{i=1}^{8} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}_q \tag{21}$$

where the sum is an unrestricted sum over the components of $\boldsymbol{n}$ and we regard the components m_i as given in terms of n_i by the TBA system (2). We split the sum into two parts with the restrictions

$$n_2 + n_4 + n_8 = \mu, 1-\mu \pmod 2 \tag{22}$$

where $\mu = 0$ and 1 gives the parity of L. After making the shifts $n_1 \to L/2 - n_1 - \ell$ and $n_1 \to L/2 - n_1 - 1/2 - \ell$, respectively, where

$$\ell = (3n_2 + 4n_3 + 5n_4 + 6n_5 + 4n_6 + 2n_7 + 3n_8)/2 \tag{23}$$

the fermionic sum can be written as

$$\sum_{\substack{\overline{n} \\ n_2+n_4+n_8=\mu \pmod 2}} q^{(2n_1)^2/2+\overline{n}^T C_{E_7}^{-1}\overline{n}} \begin{bmatrix} L/2+3(2n_1)/2-\ell \\ 2(2n_1) \end{bmatrix}_q \prod_{i=2}^{8} \begin{bmatrix} n_i+\overline{m}_i \\ n_i \end{bmatrix}_q$$

$$+ \sum_{\substack{\overline{n} \\ n_2+n_4+n_8=1-\mu \pmod 2}} q^{(2n_1+1)^2/2+\overline{n}^T C_{E_7}^{-1}\overline{n}} \begin{bmatrix} L/2+3(2n_1+1)/2-\ell \\ 2(2n_1+1) \end{bmatrix}_q \prod_{i=2}^{8} \begin{bmatrix} n_i+\overline{m}'_i \\ n_i \end{bmatrix}_q . \tag{24}$$

Here $\overline{m}_i$ and $\overline{m}'_i$ satisfy the TBA system

$$\overline{n}+\overline{m} = \tfrac{1}{2}(\mathcal{I}_{E_7}\overline{m} + (2\overline{n}_1-1)e_2 + e_i) \qquad i=2,\ldots,8 \tag{25}$$

where $\overline{n}_1 = 2n_1, 2n_1+1$ is even or odd, respectively. Combining the two sums into one sum and replacing $\overline{n}_1$ with n_1 now gives

$$\sum_{\substack{\overline{n} \\ n_1+n_2+n_4+n_8=\mu \pmod 2}} q^{n_1^2/2+\overline{n}^T C_{E_7}^{-1}\overline{n}} \begin{bmatrix} L/2+3n_1/2-\ell \\ 2n_1 \end{bmatrix}_q \prod_{i=2}^{8} \begin{bmatrix} n_i+\overline{m}_i \\ n_i \end{bmatrix}_q . \tag{26}$$

Taking the limit $L \to \infty$ and equating this to the bosonic RHS gives the identity

$$\sum_{\substack{\overline{n} \\ n_1+n_2+n_4+n_8=\mu \pmod 2}} \frac{q^{n_1^2/2}}{(q)_{2n_1}} q^{\overline{n}^T (C_{E_7})^{-1}\overline{n}} \prod_{i=2}^{8} \begin{bmatrix} n_i+\overline{m}_i \\ n_i \end{bmatrix}_q = q^{1/20} \chi^{(4)}_{1+\mu,1}(q)\chi^{(5)}_{1,1}(q) . \tag{27}$$

This identity clearly has an $A_1 \times E_7$ structure and is very similar to the E_7 identity of Kedem *et al* [9]. Indeed, these results suggest that the $c=\frac{1}{2}$ character should explicitly factor out of the LHS of this identity, but we have been unable to do this.

Some counting formulae

Lastly we list some fermionic expressions for the number of L-step paths on the dilute A_3 adjacency graph $\mathcal{G}_{dA_3}$, shown in figure 1(c). These results come about by realizing that in the critical $q \to 1$ limit, the function $B_{r,s}(L,a,b)$ counts the number of paths from a to b of length L on $\mathcal{G}_{dA_3}$ [3]. In other words, in this limit, the function $B_{r,s}(L,a,b) = (\mathcal{I}_{dA_3})^L_{a,b}$, where $\mathcal{I}_{dA_3}$ is the incidence matrix corresponding to $\mathcal{G}_{dA_3}$. (We remind the reader that $\lim_{q\to 1} \begin{bmatrix} N \\ m_1,m_2,\ldots,m_n \end{bmatrix}_q = \binom{N}{m_1,m_2,\ldots,m_n}$, the RHS being an ordinary multinomial.)

Setting $q \to 1$ in some of our 'fermionic sum = bosonic sum' identities (not all of which are listed in this paper) we find

$$\begin{aligned} F_1(L)|_{q=1} &= (\mathcal{I}^{dA_3})^L_{1,1} \qquad & F_2(L)|_{q=1} &= (\mathcal{I}^{dA_3})^L_{2,2} \\ F_7(L)|_{q=1} &= (\mathcal{I}^{dA_3})^L_{1,2} \qquad & F_8(L)|_{q=1} &= (\mathcal{I}^{dA_3})^{L+1}_{1,3} . \end{aligned} \tag{28}$$

As $\mathcal{I}_{dA_3}$ has only four distinct entries these results are complete.

Summary and discussion

In this letter we have shown directly that in the 2^+ regime the dilute A_3 model exhibits a hidden E_8 structure. This was achieved by rewriting the known bosonic expressions for the one-dimensional configuration sums in terms of fermionic sums involving the E_8 root system. Our results confirm recent work of [1] where an E_8 structure was found using the Bethe ansatz approach together with an appropriate string hypothesis. As a by-product of our work, we prove E_8 [9] and $A_1 \times E_7$-type Rogers–Ramanujan identities.

To conclude, we point out that a similar programme can be carried out for the dilute A_4 and A_6 models [2]. In doing so we find that in regime 2 these models exhibit E_7 and E_6 structures, respectively. This again confirms the earlier findings of [13] that these two models correspond to the exceptional S-matrices of Zamolodchikov and Fateev [14]. We hope to report these results together with the complete proof of identity (6) in a future publication.

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